Exercises for 'Topics in complex analysis'

(19/11/2025)

H 10.1 (On Picard's little theorem)

a) Use Proposition 6.12 to prove Picard's little theorem without using Picard's great theorem.

b) Show that Picard's little theorem is equivalent to the following statement: if $f, g : \mathbb{C} \to \mathbb{C}$ are entire functions such that $e^f + e^g = 1$, then f and g are constant.

H 10.2 (Landau's improvement of Picard's little theorem)

a) Show that there exists a function $R : \mathbb{C} \setminus \{0, 1\} \to (0, +\infty)$ such that for any $a \in \mathbb{C} \setminus \{0, 1\}$ we have

$$\left\{ f \in \mathcal{H}(\overline{B_{R(a)}(0)}) : \ f(0) = a,\ f'(0) = 1, f \text{ omits } \{0, 1\} \right\} = \emptyset.$$

Hint: Set $R(a) = 3L(\frac{1}{2}, |a|)$, where L is given by Schottky's theorem.

b) Show that the statement in a) implies Picard's little theorem.

H 10.3 (Equivalent version of Picard's great theorem)

Prove that Picard's great theorem (Theorem 6.2) is equivalent to the following statement: for any holomorphic function $f : B_1(0) \setminus \{0\} \to \mathbb{C} \setminus \{0, 1\}$, either f or $1/f$ is bounded in a neighborhood of 0.

H 10.4 (An even sharper version of Montel's theorem)

Let $G \subset \mathbb{C}$ be a simply connected domain and for $m \in \mathbb{N}$ define

$$\mathcal{F}_m := \{f : G \to \mathbb{C} \text{ holomorphic with } f(G) \cap \{0\} = \emptyset \text{ and } \#\{f = 1\} \leq m\},$$

where # denotes the cardinality of a set. Show that for any sequence $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{F}_m$, either the whole sequence $|f_n|$ converges locally uniformly to $+\infty$, or there exists a subsequence $\{f_{n_j}\}_{j\in\mathbb{N}}$ that converges locally uniformly to a holomorphic function $f : G \to \mathbb{C}$.
Hint: Consider a suitable root $\sqrt[k]{f}$ for $f \in \mathcal{F}_m$.